Homework 3, due 9/24

Only your **four** best solutions will count towards your grade.

1. Suppose that $f, g : \mathbf{C} \to \mathbf{C}$ are holomorphic, and $|f(z)| \leq |g(z)|$ for all $z \in \mathbf{C}$. Prove that $f = cg$ for some $c \in \mathbf{C}$.

2. Let $f : \mathbf{C} \to \mathbf{C}$ be holomorphic and injective. Show that $f(z) = az + b$ for some $a, b \in \mathbf{C}$, with $a \neq 0$.

3. Compute the integral
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx.$$

4. Compute the integral
$$\int_{0}^{\infty} \frac{1}{x^3+1}\, dx.$$
Hint: consider the integral of $\log z/(z^3+1)$ *along a suitable contour.*

5. Use the residue formula to prove that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$
by considering the integral of $f(z) = \frac{\cot(\pi z)}{z^2}$ around large circles centered at the origin.

6. Suppose that f is holomorphic on the punctured disk $D(0,1) \setminus \{0\}$, and that for some $A, \epsilon > 0$ we have
$$|f(z)| < A|z|^{-1+\epsilon},$$
for all $z \in D(0,1) \setminus \{0\}$. Show that 0 is a removable singularity of f.

7. Prove that the function $f(z) = \sin(1/z)$ has an essential singularity at $z = 0$.